A Homological Approach to Factorization

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JMM, Jan 2017

Acknowledgement: This presentation was supported in part by the AMS Graduate Student Travel Grant.

Motivation: Spectrum of Factorization Behavior

Def: An element $x \in D$ is (i) *atomic* if x is a product of irreducibles, (ii) *almost-atomic* if some atomic $b \in D$ exists so that $xb = a$ is atomic, (iii) *quasi-atomic* if xy is atomic for any $y \in D$.

Ex: [Almost-atomic (Coykendall, -)] Let $p \in \mathbb{Z}$ be an odd prime, $R := (\mathbb{F}_2[X, X^{2/p}, X^{2/p^2}, \ldots])_{\mathfrak{M}}$. Then each X^{2/p^m} is not atomic, X is irreducible, but $X^2 = X^{2/p^m} \cdot X^{2/p^m} \cdots X^{2/p^m}$.

Ex: [Quasi-atomic (Lebowitz-Lockard)] $R := \mathbb{Z}[X] + X^2\mathbb{R}[X]$. Non-zero non-units with order 0 or 1 are atomic, higher order elements are quasi-atomic or almost-atomic.

Examples of quasi-atomic domains that are not almost-atomic are delicate. **Wanted/Challenge:** Example of quasi-atomic quasi-local domain with a unique irreducible up to associates and is not almost-atomic.

Object of Study: Cochains of localizations

Let D be an integral domain, quotient field $D \subseteq \mathbb{F}$, unit group $U(D) \subseteq D$. The (po-)**group of divisibility** is $G(D) := \mathbb{F}^*/U(D)$. Atoms in D are atoms in $G(D)$.

Approach: Study cochains of localizations at sat'd. mult. closed sets generated by atoms

$$D \subseteq D_{S_1} \subseteq D_{S_2} \subseteq \cdots$$

Compute $G(D_{S_i})$, construct cochain complexes, compute cohomology groups, discuss properties related to factorization in D.

Why? Groups of divisibility model factorization using po-group structure. Localizing at atoms changes factorization, creates new units and irreducibles. **Ex:** $\mathbb{F}[X, Y, \frac{Y}{X}, \frac{Y}{X^2}, \ldots]$

Cochain of localizations $\to$ cochain of epimorphisms

Let $\mathcal{S}$ be all sat'd. mult. closed sets in D, let $\mathcal{H}$ be all convex directed subgroups in $G(D)$.

Question: Computing group of divisibility at each step, can we fill in the bottom row with po-group morphisms?

$$\begin{array}{ccccccc}
D & \xrightarrow{\subseteq} & D_{S_1} & \xrightarrow{\subseteq} & D_{S_2} & \xrightarrow{\subseteq} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G(D) & & G(D_{S_1}) & & G(D_{S_2}) & &
\end{array}$$

Answer: Yes, group of divisibility is functorial (Coykendall, -). Moreover, (i) $\exists$ one-to-one correspondence $\Theta : \mathcal{S} \to \mathcal{H}$ (Mott), (ii) $\exists$ $H \in \mathcal{H}$ such that $G(D_S) = G(D)/H$ (Mott), and (iii) morphisms in bottom row are natural epimorphisms.

Cochain of localizations → cochain of epimorphisms

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\downarrow & & \downarrow & & \downarrow & & \\
G(D) & \underset{\pi_0}{\twoheadrightarrow} & G(D_{S_1}) & \underset{\pi_1}{\twoheadrightarrow} & G(D_{S_2}) & \underset{\pi_2}{\twoheadrightarrow} & \cdots
\end{array}$$

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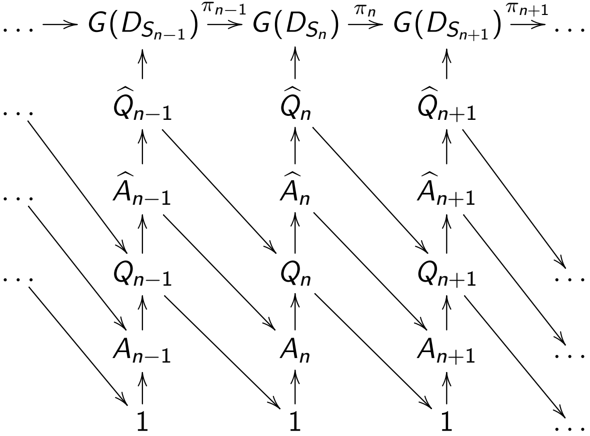
Bulding cochain complexes from cochains of epimorphisms

Define $\pi_n : G(D)/H_n \twoheadrightarrow G(D)/H_{n+1}$, $Q_n := \ker(\pi_n) = H_{n+1}/H_n$, $\widehat{Q}_n = \pi_n^{-1}(Q_{n+1})$, let $A_n \subseteq Q_n$ be any subgroup, $\widehat{A}_n = \pi_n^{-1}(A_{n+1})$.



(Rightward maps induced by $\{\pi_n\}$ restricted appropriately).

Quotient Sequence to Cochain Complexes II

Now we have cochain complexes

$$A_\bullet = \cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots$$
$$Q_\bullet = \cdots \longrightarrow Q_{n-1} \longrightarrow Q_n \longrightarrow Q_{n+1} \longrightarrow \cdots$$
$$\widehat{A}_\bullet = \cdots \longrightarrow \widehat{A}_{n-1} \longrightarrow \widehat{A}_n \longrightarrow \widehat{A}_{n+1} \longrightarrow \cdots$$
$$\widehat{Q}_\bullet = \cdots \longrightarrow \widehat{Q}_{n-1} \longrightarrow \widehat{Q}_n \longrightarrow \widehat{Q}_{n+1} \longrightarrow \cdots$$

$\widehat{Q}_\bullet$ is exact, $Q_\bullet$ and $A_\bullet$ are trivial. We also obtain quotient cochain complexes like

$$\frac{\widehat{A}_\bullet}{Q_\bullet} = \cdots \longrightarrow \frac{\widehat{A}_{n-1}}{Q_{n-1}} \longrightarrow \frac{\widehat{A}_n}{Q_n} \longrightarrow \frac{\widehat{A}_{n+1}}{Q_{n+1}} \longrightarrow \cdots$$

among others, some of which are trivial, some are exact... but the complexes $Q_\bullet$, $\widehat{A}_\bullet$, and $\widehat{A}_\bullet/Q_\bullet$ seem to play an important role.

Cochain complexes $\to$ cohomology groups

For a cochain complex $X_\bullet = \cdots \xrightarrow{\delta_{n-1}} X_n \xrightarrow{\delta_n} X_{n+1} \xrightarrow{\delta_{n+1}} \cdots$, we define the n^{th} cohomology group $H^n(G, X_\bullet) = \ker(\delta_n)/\mathrm{Im}(\delta_{n-1})$.

For a short exact sequence of cochain complexes $1 \to X_\bullet \to Y_\bullet \to Z_\bullet \to 1$, we have the long exact sequence in cohomology

$$H^0(G, X_\bullet) \to H^0(G, Y_\bullet) \to H^0(G, Z_\bullet) \to H^1(G, X_\bullet) \to \cdots$$

For $1 \to Q_\bullet \to \widehat{A}_\bullet \to \frac{\widehat{A}_\bullet}{Q_\bullet} \to 1$, we get

$$Q_0 \overset{\cong}{\to} Q_0 \overset{0}{\to} A_1 \overset{\subseteq}{\hookrightarrow} Q_1 \overset{\pi}{\twoheadrightarrow} Q_1/A_1 \overset{0}{\to} A_2 \overset{\subseteq}{\hookrightarrow} \cdots$$

which reduces to $1 \to Q_0 \cong Q_0 \to 1$ for $n = 0$ and for $n \geq 1$,

$$1 \to A_n \overset{\subseteq}{\hookrightarrow} Q_n \overset{\pi}{\twoheadrightarrow} Q_n/A_n \to 1$$

Cohomology, torsion, factorization, oh my!

How do we interpret the resulting short exact sequence?

$$1 \to A_n \to Q_n \to Q_n/A_n \to 1$$

A_n is subgroup generated by atoms, Q_n is subgroup generated by quasi-atomic elements.

Hence, Q_n/A_n is composed of the cosets of the atomic subgroup with quasi-atomic representatives! Thus, torsion in Q_n/A_n brings quasi-atomic elements under the almost-atomic umbrella.

Ex: Let $T = \mathbb{R}[X, Y^r \mid r \in \mathbb{Q}^+]$, $\mathfrak{M} \subseteq T$ the ideal of monomials, set $R' := T_{\mathfrak{M}}$, $I := (X^2 + Y^2) \subseteq R'$, $R := R'/I$. Then X is an atom, each $(Y^r U(R))A_n$ has finite order in A_n, and each Y^r is quasi-atomic and not almost-atomic.

Foundation, generalizations, further reading

Mott described the one-to-one correspondence Θ in *Convex directed subgroups of a group of divisibility* (Can. Journal of Math., 1974).

Above we glossed over many details involving po-group theory, but po-groups involve quite a bit of delicacy. Details of our approach, with some extensions (structure theorems) at `https://arxiv.org/abs/1302.4759`.

Lebowitz-Lockard presents example $\mathbb{Z}[X] + X^2\mathbb{R}[X]$ along with some further generalizations of factorization behavior at `https://arxiv.org/abs/1610.05874`.