A Functorial Look at Factorization

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Plan for the talk

Inquiry: how does factorization in general (non-atomic) integral domains work? Abstract tactic: look at problem in some category, **C**, through a functorial lens into (or out of) some better-known category, **D**. Examples: Presheaves assign topological properties to **C**, group representations place groups into **C**, etc. Our setting: map integral domains into partially ordered abelian groups.

(i) **Background**: po-groups, integral domains, factorization, sell you on the idea with the Jaffard-Kaplansky-Ohm Theorem

(ii) **Main object**: We build the group of divisibility functor $\mathfrak{G} : \mathbf{Dom} \to \mathbf{Pog}$

(iii) **Consequences**: JKO generalization, cochain complexes, their cohomology groups, what torsion means

Background I: What is the group of divisibility?

Integral domains always have a field of fractions:

$$D \subseteq \mathbb{F} = \left\{ \frac{y}{x} \mid x, y \in D, x \neq 0 \right\}$$

Divisibility in D provides a partial order on $\mathbb{F}^\times / U(D)$:

$$\frac{y}{x} \in D \cap \mathbb{F} \Leftrightarrow \overline{x} \leq \overline{y} \text{ in } \mathbb{F}^\times / U(D)$$

Example: In $\mathbb{Z} \subseteq \mathbb{Q}$, we have $\frac{1}{9} \leq \frac{2}{3}$ since $\frac{2/3}{1/9} = 6 \in \mathbb{Z}$.

Background II: Group of Div. Properties

This gives us a partially ordered abelian group $G(D) = \mathbb{F}^\times / U(D)$ (po-group) with some nice properties:

(i) Positive elements are domain elements: $\overline{1} < \overline{\frac{x}{y}} \Rightarrow \frac{x}{y} \in D$

(ii) Minimal positive/atoms: since $\overline{1} < \overline{y} \leq \overline{x}$ if and only if $y \in D$ and $y \mid x$ in $D \ldots$ irreducibles in D give atoms in $G(D)$.

(iii) Notion of convexity: $H \subseteq G$, $h_1 \leq g \leq h_2 \Rightarrow g \in H$

(iv) Maps between po-groups: given $G \to G/H$, $xH \preceq yH$ if and only if $x \leq yh$ for some $h \in H$ (reflexive, transitive, but not antisymmetric). Analogy: non-abelian setting, kernels are normal... in abelian po-groups, we will require kernels to be convex (first iso theorem!)

Background III: An o-Isomorphism Theorem

Theorem (First o-Isomorphism)

Let G_1, G_2 be (abelian) po-groups and $\phi : G_1 \to G_2$ a group homomorphism that is order-preserving ($x \leq y \Rightarrow \phi(x) \leq \phi(y)$). Then:

(i) $\ker(\phi)$ is convex subgroup of G_1 from antisymmetry,

(ii) $\phi(G_1)$ is a subgroup of G_2, and

(iii) o-isomorphism: $\phi(G_1) \simeq G_1 / \ker(\phi)$

Defs/Facts: A group homomorphism $f : G_1 \to G_2$ is an *o-homomorphism* if $x \leq y \Rightarrow f(x) \leq f(y)$. An o-hom, f, is called an *o-epimorphism* (o-epic) if $f : G \to G/H$ has convex H and $f(G^+) = (G/H)^+$. An o-epic f that is a group isomorphism is called an *o-isomorphism* (or an o-iso).

Background III: JKO Theorem

Po-groups are nice: factorization behavior in the domain is encoded in the partial order.

Theorem (Jaffard-Kaplansky-Ohm)

Any lattice-ordered group, G, is the group of divisibility of a Bézout domain, D.

Recall Bézout domains enjoy greatest common divisors that are linear combinations.

Background IV: JKO Proof Sketch (Ohm)

Assume G is lattice ordered. Then G admits a *lattice realization*: $\exists$ o-iso $\delta : G \to \bigoplus_{\lambda \in \Lambda} G_\lambda$ where (i) $\delta(a \wedge b) = \delta(a) \wedge \delta(b)$, (ii) each $\pi_\lambda \circ \delta$ is surjective, (iii) each G_λ lattice ordered .

Build a polynomial field extension $K = \mathbb{F}(\{X_g \mid g \in G\})$ and valuation maps defined on monomials

$$\begin{aligned} \omega_\lambda : & K \to G_\lambda \\ & X_g^n \mapsto n \cdot \pi_\lambda \circ \delta(g) \end{aligned}$$

Extend to polynomials: $\omega_\lambda(f(\underline{X})) = \inf \omega_\lambda(m)$ (inf taken over monomials). Extend to rationals: $\omega_\lambda(f/g) = \omega_\lambda(f) - \omega_\lambda(g)$.

Background IV: Proof sketch

Then the following map may be regarded as a semivaluation:

$$\omega(f) = \delta^{-1}\left((\omega_\lambda(f))_{\lambda\in\Lambda}\right) \in G$$

with domain $D = \{f \in K \mid \omega(f) \geq 0\}$. All that remains: D is an integral domain whose group of divisibility is o-iso to G, and it's Bézout... (omitted).

Background V: JKO Takeaway

Factorization informs po-group behavior but also vice versa.

Question: given po-group G, can we find a domain, D, such that $G = G(D)$? In general, no, not all po-groups come from groups of divisibility.

Question (JKO Extension): given a po-group, G, can we find a domain, D, such that $G \approx G(D)$ in some sense?

Main Object I: Mott's Correspondence

For a functorial look, we need a functor, so we need categories. JL Mott proved the following theorem (1973):

Theorem (Mott)

Let D be an integral domain and $G(D)$ be its group of divisibility. Let

$$\mathcal{S} = \{S \subseteq D \mid 0 \notin S, S \text{ sat'd, mult. closed}\}$$

$$\mathcal{H} = \{H \subseteq G(D) \mid H \text{ convex directed subgp}\}$$

Then there is a one-to-one correspondence, say $\Theta : \mathcal{S} \to \mathcal{H}$. Furthermore, for any $S \in \mathcal{S}$, we have the equality $G(D_S) = G(D)/\Theta(S)$.

Since $\Theta(S)$ is convex, $G(D)/\Theta(S)$ is a po-group...

Main Object II: Group of Div. as a Functor

Define **Dom**, **Pog**, $\mathfrak{G}$ like so:

(i) Objects of **Dom**: integral domains, D. Arrows: (sat'd) localizations, $D \subseteq D_S$.

(ii) Objects of **Pog**: abelian po-groups, G. Arrows: po-group o-homom. (convex kernels), $f : G_1 \to G_2$.

(iii) Define $\mathfrak{G}$: **Dom** $\to$ **Pog** by defining $\mathfrak{G}(D) = G(D)$ and defining $\mathfrak{G}(D \subseteq D_S)$ to be $\pi : G(D) \to G(D)/\Theta(S)$.

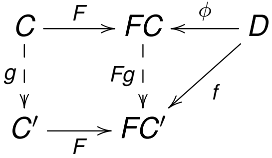
Then $\mathfrak{G}$ is a covariant functor and the following diagram commutes:

$$\begin{array}{ccc} D & \xrightarrow{\mathfrak{G}} & \mathfrak{G}(D) = G(D) \\ \Big\downarrow \subseteq & & \Big\downarrow \pi \\ D_S & \xrightarrow{\mathfrak{G}} & \mathfrak{G}(D_S) = G(D)/\Theta(S) \end{array}$$

Main Object III: Through the Functorial Looking Glass

To formalize the JKO extension question, regard the group of divisibility as a functor, $\mathfrak{G} : \mathbf{Dom} \to \mathbf{Pog}$ as above.

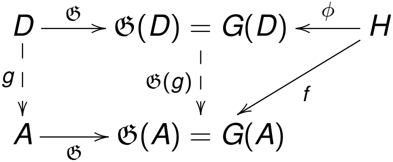
Def: Given any functor $F : \mathbf{C} \to \mathbf{D}$, an *initial morphism* from $D \in \mathbf{D}$ to F is a pair, (C, ϕ), which consists of an object $C \in \mathbf{C}$ and an arrow in $\mathbf{D}$ with the universal mapping property:



In a sense, FC is the first functorial image D is allowed to touch.

Main Object IV: Toward the Functor

Hence, an *initial morphism* from any $H \in$ **Pog** to $\mathfrak{G}$ would be a pair, (D, ϕ), which consists of a domain, $D \in$ **Dom**, and an arrow in **Pog**, say ϕ, universal in the diagram



That is: $G(D)$ is the first group of divisibility that the arbitrary po-group H can touch. Here, $g : D \to A$ is a localization, $\mathfrak{G}(g) : G(D) \to G(A)$ is o-epic. We want to find a domain, D, such that for any $f : H \to G(A)$, we can fit $D \subseteq A$ with $A = D_S$ for some S. Tall order?

Main Object V: Categorical extension of JKO

Recall: JKO Theorem establishes that lattice ordered abelian po-groups are groups of divisibility. The JKO extension question: for an arbitrary abelian po-group, G, does an initial morphism from G to $\mathfrak{G}$ always exist?

Now we have our functor, so we may be able to start answering the question.

Consequences I: JKO extension probably false in general

Lemma: If initial morphisms always exist to a functor, such as $\mathfrak{G}$, then $\mathfrak{G}$ is a right-adjoint ($\exists \mathfrak{D}$: **Pog** $\to$ **Dom** such that $\mathfrak{D} \dashv \mathfrak{G}$).

But this is almost surely too much to ask for on the unrestricted categories.

Certainly $\mathfrak{G}$ restricted to the categories of Bézout domains and lattice-ordered po-groups has an adjoint (corollary of the JKO theorem). JKO question starts to get pretty abstract: which subcategories of **Dom** have initial morphisms to $\mathfrak{G}$? (No answers yet...)

Onto some concrete results from $\mathfrak{G}$.

Consequences II: Towers of Localizations

What else can we do with the functor? Towers of localizations turn into projections

$$\begin{array}{ccccccc}
D & \xrightarrow{\subseteq} & D_{S_1} & \xrightarrow{\subseteq} & D_{S_2} & \xrightarrow{\subseteq} & \cdots \\
\downarrow{\scriptstyle \mathfrak{G}} & & \downarrow{\scriptstyle \mathfrak{G}} & & \downarrow{\scriptstyle \mathfrak{G}} & & \\
G(D) & \xrightarrow{\pi_0} & G(D)/\Theta(S_1) & \xrightarrow{\pi_1} & G(D)/\Theta(S_2) & \xrightarrow{\pi_2} & \cdots
\end{array}$$

We can very generally convert (bottom) surjections into a multitude of cochain complexes by including kernels in the diagram and their inverse images... but...

Consequences III: Choice of localization

Before applying the above, where are we localizing?

Ex: Let R be a 2-dimensional discrete valuation domain with prime spectrum $0 \subseteq \mathfrak{p} \subseteq \mathfrak{m} \subseteq R$.

(i) The maximal ideal is principal, say $\mathfrak{m} = (x)$.

(ii) The element x is a unique irreducible (up to units).

(iii) Every nonzero nonunit factors into a power of x and another element.

(iv) No element of $\mathfrak{p}$ is a product of irreducibles.

(v) Elements in $\mathfrak{m} \setminus \mathfrak{p}$ are atomic (products of x).

(vi) Localizing at $\mathfrak{p}$ yields a PID, localizing at 0 yields the quotient field.

Just weird! If we remove the non-atoms, we get a field, i.e. no factorization information. So we localize at atoms.

Consequences IV: Localizing at atoms

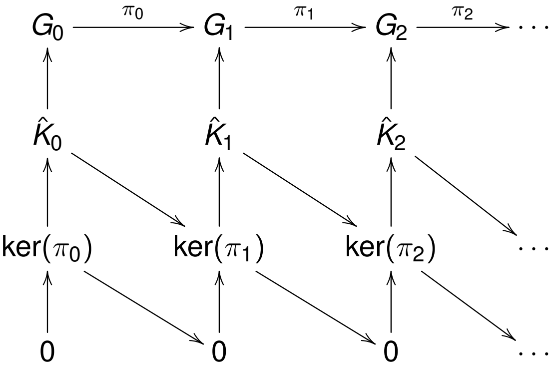
If we localize at saturated multiplicatively closed subsets generated by irreducibles, we are projecting onto quotient groups with $\ker(\pi)$ convex, directed, generated by atoms.

So if A is the subgroup generated by atoms and $\ker(\pi)$ is the smallest convex subgroup containing A, then we get

$$G_0 \to G_1 = G_0/\ker(\pi_0) \to G_2 = G_0/\ker(\pi_1) \to \cdots$$

How to construct cochain complexes?

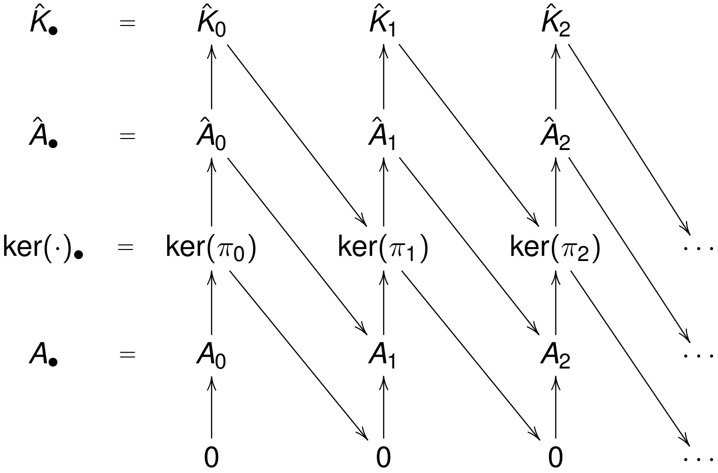
Consequences IV: Build cochain complexes



Throw in kernels and inverse images of kernels

Consequences V: Structure in the kernels...

Throw in substructures $0 \subseteq A_i \subseteq \ker \pi_i$:



Consequences V: One non-trivial, non-exact complex

We get the (exact) cochain complex

$$\hat{K}_\bullet = \hat{K}_0 \to \hat{K}_1 \to \cdots$$

and the (trivial) complexes

$$\ker(\cdot)_\bullet = \ker(\pi_0) \to \ker(\pi_1) \to \cdots$$

$$A_\bullet = A_0 \to A_1 \to \cdots$$

And the generally nontrivial, nonexact

$$\hat{A}_\bullet = \hat{A}_0 \to \hat{A}_1 \to \cdots$$

which has n^{th} cohomology group $\ker(\pi_n)/A_n$. *More structure in* $\ker(\pi_n)$ *gives more cochain complexes.*

Consequences VII: So what is torsion?

Every $\ker(\pi_i)$ is the convex directed subgroup generated by atoms. Kernels contain $A_i \subseteq \ker(\pi_i)$ (directed subgroup generated atoms). This gives us the n^{th} cohomology group $\ker(\pi_n)/A_n$.

Mott's correspondence: $\ker(\pi) \subseteq G(D)$ corresponds to the saturated multiplicatively closed set generated by the irreducibles. **Similarly:** A_n corresponds to the multiplicatively closed set generated by the irreducibles, need not be saturated.

Consequently, *torsion* (group) elements of the cohomology correspond to *non-atomic divisors of atomic ring elements:* elements in the purgatory between atomic and antimatter.

(Some) Further Routes of Inquiry

(i) Expanding Bézout domains to the largest subcategory of **Dom** that admits a left adjoint for $\mathfrak{G}$.
(ii) Structure theorems, sequence of group projections.
(iii) Examining the cohomological consequences of a more rich structure in the kernels.
(iv) Groups of divisibility: intersection of partial orders and group theory. Can these techniques be meaningfully extended to other famous po-groups (algebraic K-theory, etc.)?
(v) Factorization and topology: connections between partially ordered topological spaces and po-group theory, Haar measures.
(vi) Orders and relations define graphs: graph-theoretic consequences for factorization? Factorization consequences for graph theory?