# Antimatter v. Quasi-Atomic Domains

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# Tale of Two... -cities

For an arbitrary ring, can we assess/quantify/characterize the lack of *atomicity*?

- Atomic integral domain, $R$: write each (nonzero, nonunit) $x \in R$ as $x = \prod_n p_n$ where each $p_n \in \text{Irr}(R)$
- Antimatter integral domain: $\text{Irr}(R) = \emptyset$
- Mixed behaviors: some $x \in R$ factor into irreducibles, some do not.

In this talk: the spectrum of factorization behavior, defining quasi-atomic domains, constructing provable examples.

# Spectrum of Factorization



Any field is vacuously atomic and antimatter.

More interesting: UFDs, HFDs, Noetherian domains, ACCP domains are atomic

For a less trivial antimatter domain, let $\mathbb{F}$ be any field and $X = \{x^q \mid q \in \mathbb{Q}, q > 0\}$, and let $\mathfrak{m} = (X)$. Then $R = \mathbb{F}[X]_{\mathfrak{m}}$ is antimatter:

- $f(x) \in U(R) \Leftrightarrow f(0) \neq 0$ (so $x^q$ is nonzero nonunit)
- Every nonzero nonunit can be written $f(x)x^q$ for some unit $f(x)$ (pull out minimal power in $x$)
- Since $x^q = x^{q/2}x^{q/2}$, every element is reducible.

# Hold on a minute...



This last example has exponents in $\mathbb{Q}^+$. What about other additive submonoids of $\mathbb{Q}^+$?

**Ex** (Coykendall, -): Let $M = \langle 1, \frac{2}{3}, \frac{2}{3^2}, \frac{2}{3^3}, \cdots \rangle$. Let $X = \{x^m \mid m \in M\}$ and $R = \mathbb{F}_2[X]_\mathfrak{m}$:

- $f(x) \in U(R) \Leftrightarrow f(0) \neq 0$,
- $x$ is irreducible since $1 \in M$ cannot be written $1 = \sum_{k=1}^{N} \frac{2a_k}{3^k}$ with each $0 \leq a_k \leq 2$, and
- We have $x^2 = x \cdot x = \underbrace{x^{2/3} \cdot x^{2/3} \cdot x^{2/3}}$

# A Generalization of Atomicity



In $R = \mathbb{F}_2[X]_{\mathfrak{m}}$, $x$ is the only irreducible *monomial*, but elements of the form $x + x^{2/3^n}$ are also irreducible.

Note $x^{2/3}$ is not atomic but there exists an atomic $a$ such that $ax^{2/3}$ is atomic:

$$x^{2/3}\underbrace{(x + x^{2/3})}_{\mathrm{Irr}(R)}\underbrace{(x + x^{2/3})}_{\mathrm{Irr}(R)} = \underbrace{(x^2)}_{\mathrm{Irr}(R)}\underbrace{(1 + x^{2/3})}_{U(R)} = ux^2$$

**In fact**: Every monomial, $x^m \in R$ is *almost atomic* in this sense.

**In fact:** Any $f(x) \in R$ is irreducible if and only if $f(x) \sim x$ or $f(x) \sim x + vx^{2/3^n}$ for some $v \in U(R)$, $n \geq 1$

**Def:** A *numerical semigroup* is a sub-semigroup of $\mathbb{N}$, $A \subseteq \mathbb{N}$, such that $|\mathbb{N} \setminus A|$ is finite.

**Thm** (Gotti, 2016): A monoid, $M \subseteq \mathbb{Q}^+$, satisfying $(\star_1)$ and $(\star_2)$, is atomic if and only if it $M \setminus 0$ is isomorphic to a numerical semigroup.

$(\star_1)$ Numerator set is bounded
$(\star_2)$ Only a finite subset of primes allowable in denominators

**Thm** (Gotti, 2016): If a monoid, $M$, satisfies $(\star_2)$ and $(\star_3)$, then $M$ is antimatter.

$(\star_3)$ There exists a sequence $(b_1, b_2, \ldots) \subseteq \mathbb{N}$ such that each $1/b_n \in M$ and the sequence $(v_p(b_1), v_p(b_2), \ldots)$ is strictly increasing ($p$-adic valuations with $p$ as in $\star_2$).

# Quasi-atomic defined:



This ring $R = \mathbb{F}_2[X]_{\mathfrak{m}}$ where $X = \{x^m \mid m \in M\}$ with $M = \langle 1, \frac{2}{3}, \frac{2}{3^2}, \ldots \rangle$ inspired the following definitions:

**Def**: (Boynton and Coykendall, 2013) Given a ring, $R$, we say nonzero nonunit $x \in R$ is *almost atomic* if there exists an atomic $a$ such that $ax$ is atomic. We say $x \in R$ is *quasi-atomic* if there exists an arbitrary $y \in R$ such that $xy$ is atomic.

**But what about quasi-atomic?** Hard to produce *simple* examples of quasi-atomic domains that are not almost atomic without real numbers... example: $\mathbb{Z}[X] + X^2\mathbb{R}[X]$

# Quasi-Atomic Domains



**Ex:** (Coykendall) Let $y, x_1, x_2, \ldots$ be indeterminate over $\mathbb{F}_2$, and

$$X = \left\{ y^2, x_1, x_2, \ldots, \frac{y^2}{x_1^n}, \frac{x_1}{x_2^n}, \frac{x_2}{x_3^n}, \ldots \mid n \in \mathbb{N} \right\}$$

Let $R = \mathbb{F}_2 [X]_{\mathfrak{m}}$ as before, where $\mathfrak{m}$ is the ideal generated by all monomials. Then (i) $R[y]$ is quasi-atomic, (ii) every irreducible is associate to $y$, and (iii) $\dim(R[y]) = +\infty$.

**Proof Sketch:** For (iii), we have $(y) \subsetneq (x_1) \subsetneq (x_2) \subsetneq \cdots$. Assuming (ii), every monomial divides $y^2$. Proof of (i): prove every nonzero nonunit is associate to $y + a$ for some nonunit $a \in R$, prove irreducible iff $a = 0$.

# Constructing “Smaller” Qausi-Atomic Domains



**Thm**: Let $R$ be a CK domain and not a PID with $\text{char}(R) = p > 0$ and quotient field $\mathbb{F}$. If every irreducible, $\xi \in \text{Irr}(R)$, satisfies $\xi^{1/p} \notin \mathbb{F}$, and if there exists an irreducible, $\pi \in \text{Irr}(R)$, such that $\pi^p$ factors uniquely, then there exists a 1-dimensional quasi-atomic domain with a unique irreducible element.

**Proof**: Take direct limit of a chain of polynomial extensions of the form $R \subseteq R[X_1] \subseteq R[X_1, X_2] \subseteq \cdots$, where each $X_i = \{\xi^{1/p} \mid \xi \not\sim \pi\}$

**To construct** a 1-dimensional quasi-atomic domain with a unique irreducible, it is sufficient to find a **seed domain:** need a CK domain satisfying these theorem hypotheses...